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TOWARD A GEOMETRIC ANALOGUE OF DIRICHLET'S UNIT THEOREM

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Abstract. In this article, we propose a geometric analogue of Dirichlet's unit theorem on arithmetic varieties [18], that is, if X is a normal projective variety over a finite field and D is a pseudo-effective $\mathbb{Q}$-Cartier divisor on X, does it follow that D is $\mathbb{Q}$-effective? We also give affirmative answers on an abelian variety and a projective bundle over a curve.

Introduction

Let K be a number field and O_K the ring of integers in K. Let $K(\mathbb{C})$ be the set of all embeddings $K \hookrightarrow \mathbb{C}$. For $\sigma \in K(\mathbb{C})$, the complex conjugation of σ is denoted by $\overline{\sigma}$, that is, $\overline{\sigma}(x) = \overline{\sigma(x)}$ $(x \in K)$. Here we define Ξ_K and Ξ_K^0 to be

$$\begin{cases} \Xi_K := \left\{ \xi \in \mathbb{R}^{K(\mathbb{C})} \mid \xi(\sigma) = \xi(\overline{\sigma}) \ (\forall \sigma) \right\}, \\ \Xi_K^0 := \left\{ \xi \in \Xi_K \mid \sum_{\sigma \in K(\mathbb{C})} \xi(\sigma) = 0 \right\}. \end{cases}$$

The Dirichlet unit theorem asserts that the group $O_K^\times$ consisting of units in O_K is a finitely generated abelian group of rank $s := \dim_{\mathbb{R}} \Xi_K^0$.

Let us consider the homomorphism $L : K^\times \to \mathbb{R}^{K(\mathbb{C})}$ given by

$$L(x)(\sigma) := \log |\sigma(x)| \quad (x \in K^\times,\ \sigma \in K(\mathbb{C})).$$

It is easy to see the following:

(i) For a compact set B in $\mathbb{R}^{K(\mathbb{C})}$, the set $\{x \in O_K^\times \mid L(x) \in B\}$ is finite.
(ii) $L : K^\times \to \mathbb{R}^{K(\mathbb{C})}$ extends to $L_{\mathbb{R}} : K^\times \otimes \mathbb{R} \to \mathbb{R}^{K(\mathbb{C})}$.
(iii) $L_{\mathbb{R}} : O_K^\times \otimes \mathbb{R} \to \mathbb{R}^{K(\mathbb{C})}$ is injective.
(iv) $L_{\mathbb{R}}(O_K^\times \otimes \mathbb{R}) \subseteq \Xi_K^0$.

By using (i) and (iii), we can see that $O_K^\times$ is a finitely generated abelian group. The most essential part of the Dirichlet unit theorem is to show that $O_K^\times$ is of rank s, which is equivalent to see that, for any $\xi \in \Xi_K^0$, there is $x \in O_K^\times \otimes \mathbb{R}$ with $L_{\mathbb{R}}(x) = \xi$.

In order to understand the equality $L_{\mathbb{R}}(x) = \xi$ in terms of Arakelov geometry, let us introduce several notations for arithmetic divisors on the arithmetic curve $\operatorname{Spec}(O_K)$. An arithmetic $\mathbb{R}$-divisor on $\operatorname{Spec}(O_K)$ is a pair (D, ξ) consisting of an $\mathbb{R}$-divisor D on $\operatorname{Spec}(O_K)$ and $\xi \in \Xi_K$. We often denote the pair (D, ξ) by $\overline{D}$. The

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arithmetic principal $\mathbb{R}$-divisor $\widehat{(x)}_{\mathbb{R}}$ of $x \in K^{\times} \otimes \mathbb{R}$ is the arithmetic $\mathbb{R}$-divisor given by

$$\widehat{(x)}_{\mathbb{R}} := \left(\textstyle\sum_P \operatorname{ord}_P(x)[P], -2L_{\mathbb{R}}(x)\right),$$

where P runs over the set of all maximal ideals of O_K and

$$\operatorname{ord}_P(x) := a_1 \operatorname{ord}_P(x_1) + \cdots + a_r \operatorname{ord}_P(x_r)$$

for $x = x_1^{a_1} \cdots x_r^{a_r}$ ($x_1, \ldots, x_r \in K^{\times}$ and $a_1, \ldots, a_r \in \mathbb{R}$). The arithmetic degree $\widehat{\deg}(\overline{D})$ of an arithmetic $\mathbb{R}$-divisor $\overline{D} = (\sum_P a_P[P], \xi)$ is defined to be

$$\widehat{\deg}(\overline{D}) := \sum_P a_P \log \#(O_K/P) + \frac{1}{2} \sum_{\sigma \in K(\mathbb{C})} \xi(\sigma).$$

Note that

$$\widehat{\deg}\left(\widehat{(x)}_{\mathbb{R}}\right) = 0 \quad (x \in K^{\times} \otimes \mathbb{R})$$

by virtue of the product formula. Further, $\overline{D} = (\sum_P a_P[P], \xi)$ is said to be effective if $a_P \geq 0$ for all P and $\xi(\sigma) \geq 0$ for all σ.

In [18, SubSection 3.4], we proved the following:

(0.1) "If $\widehat{\deg}(\overline{D}) \geq 0$, then $\overline{D} + \widehat{(x)}_{\mathbb{R}}$ is effective for some $x \in K^{\times} \otimes \mathbb{R}$."

This implies the essential part of the Dirichlet unit theorem. Indeed, we set $\overline{D} = (0, \xi)$ for $\xi \in \Xi_K^0$. As $\widehat{\deg}(\overline{D}) = 0$, by the assertion (0.1), $\overline{D} + \widehat{(y)}_{\mathbb{R}}$ is effective for some $y \in K^{\times} \otimes \mathbb{R}$, and hence $\overline{D} + \widehat{(y)}_{\mathbb{R}} = (0,0)$ because $\widehat{\deg}(\overline{D} + \widehat{(y)}_{\mathbb{R}}) = 0$. Here we set $y = u_1^{a_1} \cdots u_r^{a_r}$ such that $u_1, \ldots, u_r \in K^{\times}$, $a_1, \ldots, a_r \in \mathbb{R}$ and $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. By using the linear independency of $a_1, \ldots, a_r$ over $\mathbb{Q}$, $\operatorname{ord}_P(y) = 0$ implies $\operatorname{ord}_P(u_i) = 0$ for all maximal ideals P of O_K and $i = 1, \ldots, r$, that is, $u_i \in O_K^{\times}$ for $i = 1, \ldots, r$. Therefore, $\xi = L_{\mathbb{R}}(y^2)$ and $y \in O_K^{\times} \otimes \mathbb{R}$, as required. In this sense, the above property (0.1) is an Arakelov theoretic interpretation of the classical Dirichlet unit theorem.

In [18] and [19], we considered a higher dimensional analogue of (0.1). In the higher dimensional case, the condition "$\widehat{\deg}(\overline{D}) \geq 0$" should be replaced by the pseudo-effectivity of $\overline{D}$. Of course, this analogue is not true in general (cf. [5]). It is however a very interesting problem to find a sufficient condition for the existence of an arithmetic small $\mathbb{R}$-section, that is, an element x such that

$$x = x_1^{a_1} \cdots x_r^{a_r} \quad (x_1, \ldots, x_r \text{ are rational functions and } a_1, \ldots, a_r \in \mathbb{R})$$

and $\overline{D} + \widehat{(x)}_{\mathbb{R}}$ is effective. For example, in [18] and [19], we proved that if D is numerically trivial and $\overline{D}$ is pseudo-effective, then $\overline{D}$ has an arithmetic small $\mathbb{R}$-section. In this paper, we would like to consider a geometric analogue of the Dirichlet unit theorem in the above sense.

Let X be a normal projective variety over an algebraically closed field k. Let $\operatorname{Div}(X)$ denote the group of Cartier divisors on X. Let $\mathbb{K}$ be either the field $\mathbb{Q}$ of rational numbers or the field $\mathbb{R}$ of real numbers. We define $\operatorname{Div}(X)_{\mathbb{K}}$ to be $\operatorname{Div}(X)_{\mathbb{K}} := \operatorname{Div}(X) \otimes_{\mathbb{Z}} \mathbb{K}$, whose element is called a $\mathbb{K}$-*Cartier divisor* on X. For $\mathbb{K}$-Cartier divisors D_1 and D_2, we say that D_1 is $\mathbb{K}$-*linearly equivalent* to D_2, which is denoted by $D_1 \sim_{\mathbb{K}} D_2$, if there are non-zero rational functions $\phi_1, \ldots, \phi_r$ on X and $a_1, \ldots, a_r \in \mathbb{K}$ such that

$$D_1 - D_2 = a_1(\phi_1) + \cdots + a_r(\phi_r).$$

Let D be a $\mathbb{K}$-Cartier divisor on X. We say that D is *big* if there is an ample $\mathbb{Q}$-Cartier divisor A on X such that $D - A$ is $\mathbb{K}$-linearly equivalent to an effective $\mathbb{K}$-Cartier divisor. Further, D is said to be *pseudo-effective* if $D + B$ is big for any big $\mathbb{K}$-Cartier divisor B on X. Note that if D is $\mathbb{K}$-*effective* (i.e. D is $\mathbb{K}$-linearly equivalent to an effective $\mathbb{K}$-Cartier divisor), then D is pseudo-effective. The converse of the above statement holds on toric varieties (for example, [4, Proposition 4.9]). However, it is not true in general. In the case where k is uncountable (for example, $k = \mathbb{C}$), several examples are known such as non-torsion numerically trivial invertible sheaves and Mumford's example on a minimal ruled surface (cf. [8, Chapter 1, Example 10.6] and [14]). Nevertheless, we would like to propose the following question:

Question 0.2 ($\mathbb{K}$-version)**.** We assume that k is an algebraic closure of a finite field. If a $\mathbb{K}$-Cartier divisor D on X is pseudo-effective, does it follow that D is $\mathbb{K}$-effective?

This question is a geometric analogue of the fundamental question introduced in [18]. In this sense, it turns out to be a geometric Dirichlet's unit theorem if it is true, so that we often say that a $\mathbb{K}$-Cartier divisor D has *the Dirichlet property* if D is $\mathbb{K}$-effective. Note that the $\mathbb{R}$-version implies the $\mathbb{Q}$-version (cf. Proposition 1.5). Moreover, the $\mathbb{R}$-version does not hold in general. In Example 3.2, we give an example, so that, for the $\mathbb{R}$-version, the question should be

"Under what conditions does it follow that D is $\mathbb{K}$-effective?".

Further, the $\mathbb{Q}$-version implies the following question due to Keel (cf. [10, Question 0.9] and Remark 2.4). The similar arguments on an algebraic surface are discussed in the recent article by Langer [12, Conjecture 1.7$\sim$1.9 and Lemma 1.10].

Question 0.3 (S. Keel)**.** We assume that k is an algebraic closure of a finite field and X is an algebraic surface over k. Let D be a Cartier divisor on X. If $(D \cdot C) > 0$ for all irreducible curves C on X, is D ample?

By virtue of the Zariski decomposition, Question 0.2 on an algebraic surface is equivalent to ask the following:

"If D is nef, then is D $\mathbb{K}$-effective?".

One might expect that D is semiample (cf. [10, Question 0.8.2]). However, Totaro [24, Theorem 6.1] found a Cartier divisor D on an algebraic surface over a finite field such that D is nef but not semiample. Totaro's example does not give a counter example of our question because we assert only the $\mathbb{Q}$-effectivity in Question 0.2.

Inspired by the paper [3] due to Biswas and Subramanian, we have the following partial answer to the above question.

Theorem 0.4. *We assume that k is an algebraic closure of a finite field. Let C be a smooth projective curve over k and let E be a locally free sheaf of rank r on C. Let $\mathbb{P}(E)$ be the projective bundle of E, that is, $\mathbb{P}(E) := \operatorname{Proj}\left(\bigoplus_{m=0}^{\infty}\operatorname{Sym}^m(E)\right)$. If D is a pseudo-effective $\mathbb{K}$-Cartier divisor on $\mathbb{P}(E)$, then D is $\mathbb{K}$-effective.*

In addition to the above result, we can also give an affirmative answer for the $\mathbb{Q}$-version of Question 0.2 on abelian varieties.

Proposition 0.5. *We assume that k is an algebraic closure of a finite field. Let A be an abelian variety over k. If D is a pseudo-effective $\mathbb{Q}$-Cartier divisor on A, then D is $\mathbb{Q}$-effective.*

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1. Preliminaries

Let k be an algebraic closed field. Let C be a smooth projective curve over k and let E be a locally free sheaf of rank r on C. The projective bundle $\mathbb{P}(E)$ of E is given by

$$\mathbb{P}(E) := \operatorname{Proj}\left(\bigoplus_{m=0}^{\infty}\operatorname{Sym}^m(E)\right).$$

The canonical morphism $\mathbb{P}(E) \to C$ is denoted by f_E. A tautological divisor Θ_E on $\mathbb{P}(E)$ is a Cartier divisor on $\mathbb{P}(E)$ such that $\mathcal{O}_{\mathbb{P}(E)}(\Theta_E)$ is isomorphic to the tautological invertible sheaf $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$. We say that E is *strongly semistable* if, for any surjective morphism $\pi : C' \to C$ of smooth projective curves, $\pi^*(E)$ is semistable. By definition, if E is strongly semistable and $\pi : C' \to C$ is a surjective morphism of smooth projective curves over k, then $\pi^*(E)$ is also strongly semistable. A filtration

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{s-1} \subsetneq E_s = E$$

of E is called the *strong Harder-Narasimham filtration* if

$$\mu(E_1/E_0) > \mu(E_2/E_1) > \cdots > \mu(E_{s-1}/E_{s-2}) > \mu(E_s/E_{s-1})$$

and E_i/E_{i-1} is a strongly semistable locally free sheaf on C for each $i = 1, \ldots, s$. Recall the following well-known facts (F1)–(F5) on strong semistability.

(F1) A locally free sheaf E on C is strong semistable if and only if $\Theta_E - f_E^*(\xi_E/r)$ is nef, where ξ_E is a Cartier divisor on C with $\mathcal{O}_C(\xi_E) \simeq \det(E)$ (for example, see [16, Proposition 7.1, (3)]).

(F2) Let $\pi : C' \to C$ be a surjective morphism of smooth projective curves over k such that the function field of C' is a separable extension field over the function field of C. If E is semistable, then $\pi^*(E)$ is also semistable (for example, see [16, Proposition 7.1, (1)]). In particular, if $\operatorname{char}(k) = 0$, then E is strongly semistable if and only if E is semistable. Moreover, in the case where $\operatorname{char}(k) > 0$, E is strongly semistable if and only if $(F^m)^*(E)$

is semistable for all $m \geq 0$, where $F : C \to C$ is the absolute Frobenius map and

$$F^m = \overbrace{F \circ \cdots \circ F}^{m}.$$

(F3) If E and G are strongly semistable locally free sheaves on C, then $\mathrm{Sym}^m(E)$ and $E \otimes G$ are also strongly semistable for all $m \geq 1$ (for example, see [16, Theorem 7.2 and Corollary 7.3]).

(F4) There is a surjective morphism $\pi : C' \to C$ of smooth projective curves over k such that $\pi^*(E)$ has the strong Harder-Narasimham filtration (cf. [11, Theorem 7.2]).

(F5) We assume that k is an algebraic closure of a finite field. If E is a strongly semistable locally free sheaf on C with $\det(E) \simeq \mathcal{O}_C$, then there is a surjective morphism $\pi : C' \to C$ of smooth projective curves over k such that $\pi^*(E) \simeq \mathcal{O}_{C'}^{\oplus \operatorname{rk} E}$ (cf. [1, p. 557], [23, Theorem 3.2] and [3]).

The purpose of this section is to prove the following characterizations of pseudo-effective $\mathbb{R}$-Cartier divisors and nef $\mathbb{R}$-Cartier divisors on $\mathbb{P}(E)$. This is essentially due to Nakayama [22, Lemma 3.7] in which he works over the complex number field.

Proposition 1.1. *We assume that E has the strong Harder-Narasimham filtration:*

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{s-1} \subsetneq E_s = E.$$

Then, for an $\mathbb{R}$-divisor A on C, we have the following:

(1) *$\Theta_E - f^*(A)$ is pseudo-effective if and only if $\deg(A) \leq \mu(E_1)$.*
(2) *$\Theta_E - f^*(A)$ is nef if and only if $\deg(A) \leq \mu(E/E_{s-1})$.*

Let us begin with the following lemma.

Lemma 1.2. *We assume that E has a filtration*

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{s-1} \subsetneq E_s = E$$

such that E_i/E_{i-1} is a strongly semistable locally free sheaf on C and $\deg(E_i/E_{i-1}) < 0$ for all $i = 1, \ldots, s$. Then, $H^0(C, \mathrm{Sym}^m(E) \otimes G) = 0$ for $m \geq 1$ and a strongly semistable locally free sheaf G on C with $\deg(G) \leq 0$.

Proof. We prove it by induction on s. In the case where $s = 1$, E is strongly semistable and $\deg(E) < 0$, so that $\mathrm{Sym}^m(E) \otimes G$ is also strongly semistable by (F3) and

$$\deg(\mathrm{Sym}^m(E) \otimes G) < 0.$$

Therefore, $H^0(C, \mathrm{Sym}^m(E) \otimes G) = 0$.

Here we assume that $s > 1$. Let us consider an exact sequence

$$0 \to E_{s-1} \to E \to E/E_{s-1} \to 0.$$

By [9, Chapter II, Exercise 5.16, (c)], there is a filtration

$$\mathrm{Sym}^m(E) = F^0 \supsetneq F^1 \supsetneq \cdots \supsetneq F^m \supsetneq F^{m+1} = 0$$

such that

$$F^j/F^{j+1} \simeq \mathrm{Sym}^j(E_{s-1}) \otimes \mathrm{Sym}^{m-j}(E/E_{s-1})$$

for each $j = 0, \dots, m$. By using the hypothesis of induction,

$$H^0(C, (F^j/F^{j+1}) \otimes G) = 0$$

for $j = 1, \dots, m$ because $\mathrm{Sym}^{m-j}(E/E_{s-1}) \otimes G$ is strongly semistable by (F3) and

$$\deg(\mathrm{Sym}^{m-j}(E/E_{s-1}) \otimes G) \leq 0.$$

Moreover, since $\mathrm{Sym}^m(E/E_{s-1}) \otimes G$ is strongly semistable by (F3) and

$$\deg(\mathrm{Sym}^m(E/E_{s-1}) \otimes G) < 0,$$

we have

$$H^0(C, (F^0/F^1) \otimes G) = H^0(C, \mathrm{Sym}^m(E/E_{s-1}) \otimes G) = 0.$$

Therefore, by using an exact sequence

$$0 \to F^{j+1} \otimes G \to F^j \otimes G \to (F^j/F^{j+1}) \otimes G \to 0,$$

we have

$$H^0(C, F^{j+1} \otimes G) \xrightarrow{\sim} H^0(C, F^j \otimes G)$$

for $j = 0, \dots, m$, which implies that $H^0(C, \mathrm{Sym}^m(E) \otimes G) = 0$, as required. $\square$

Proof of Proposition 1.1. It is sufficient to show the following:

(a) If A is a $\mathbb{Q}$-Cartier divisor and $\deg(A) < \mu(E_1)$, then $\Theta_E - f^*(A)$ is $\mathbb{Q}$-effective.
(b) If A is a $\mathbb{Q}$-Cartier divisor and $\deg(A) > \mu(E_1)$, then $\Theta_E - f^*(A)$ is not pseudo-effective.
(c) If $\Theta_E - f^*(A)$ is nef, then $\deg(A) \leq \mu(E/E_{s-1})$.
(d) If $\Theta_E - f^*(A)$ is not nef, then $\deg(A) > \mu(E/E_{s-1})$.

(a) Let θ be a divisor on C with $\deg(\theta) = 1$. As E_1 is strongly semistable, by (F1), $\Theta_{E_1} - \mu(E_1) f_{E_1}^*(\theta)$ is nef, so that we can see that $\Theta_{E_1} - f_{E_1}^*(A)$ is nef and big because

$$\Theta_{E_1} - \deg(A) f_{E_1}^*(\theta) = \Theta_{E_1} - \mu(E_1) f_{E_1}^*(\theta) + (\mu(E_1) - \deg(A)) f_{E_1}^*(\theta).$$

Therefore, there is a positive integer m_1 such that $m_1 A$ is a divisor on C and

$$H^0\left(\mathbb{P}(E_1), \mathcal{O}_{\mathbb{P}(E_1)}(m_1 \Theta_{E_1} - f_{E_1}^*(m_1 A))\right) \neq 0.$$

In addition,

$$\begin{aligned} H^0\left(\mathbb{P}(E_1), \mathcal{O}_{\mathbb{P}(E_1)}(m_1 \Theta_{E_1} - f_{E_1}^*(m_1 A))\right) &= H^0(C, \mathrm{Sym}^{m_1}(E_1) \otimes \mathcal{O}_C(-m_1 A)) \\ &\subseteq H^0(C, \mathrm{Sym}^{m_1}(E) \otimes \mathcal{O}_C(-m_1 A)) \\ &= H^0\left(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m_1 \Theta_E - f_E^*(m_1 A))\right), \end{aligned}$$

so that $\Theta_E - f_E^*(A)$ is $\mathbb{Q}$-effective.

(b) Let B be an ample $\mathbb{Q}$-divisor on C with $\deg(B) < \deg(A) - \mu(E_1)$. Let $\pi : C' \to C$ be a surjective morphism of smooth projective curves over k such that $\pi^*(-A+B)$ is a Cartier divisor on C'. Note that

$$\mu\left(\pi^*(E_i/E_{i-1}) \otimes \mathcal{O}_{C'}(\pi^*(-A+B))\right) < 0$$

for $i = 1, \ldots, s$, and hence, by Lemma 1.2,

$$H^0\left(C', \mathrm{Sym}^m(\pi^*(E)) \otimes \mathcal{O}_{C'}(m\pi^*(-A+B))\right) = 0$$

for all $m \geq 1$. In particular, if b is a positive integer such that $b(-A+B)$ is a Cartier divisor, then

$$H^0\left(C, \mathrm{Sym}^{mb}(E) \otimes \mathcal{O}_C(mb(-A+B))\right) = 0$$

for $m \geq 1$. Here we assume that $\Theta_E - f_E^*(A)$ is pseudo-effective. Let a be a positive integer such that $\Theta_E - f_E^*(A) + af_E^*(B)$ is ample. Then

$$(a-1)(\Theta_E - f_E^*(A)) + \Theta_E - f_E^*(A) + af_E^*(B) = a(\Theta_E + f_E^*(-A+B))$$

is big, so that we can find a positive integer m_1 such that

$$\begin{aligned} H^0&\left(C, \mathrm{Sym}^{m_1ab}(E) \otimes \mathcal{O}_C(m_1ab(-A+B))\right) \\ &= H^0\left(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m_1ab(\Theta_E + f_E^*(-A+B)))\right) \neq 0, \end{aligned}$$

which is a contradiction.

(c) Note that

$$\mathbb{P}(E/E_{s-1}) \subseteq \mathbb{P}(E), \quad \Theta_{E/E_{s-1}} \sim \Theta_E|_{\mathbb{P}(E/E_{s-1})} \quad \text{and} \quad f_{E/E_{s-1}} = f_E|_{\mathbb{P}(E/E_{s-1})},$$

so that $\Theta_{E/E_{s-1}} - f_{E/E_{s-1}}^*(A)$ is nef on $\mathbb{P}(E/E_{s-1})$. Let $\xi_{E/E_{s-1}}$ be a Cartier divisor on C with $\mathcal{O}_C(\xi_{E/E_{s-1}}) \simeq \det(E/E_{s-1})$. If we set $e = \operatorname{rk} E/E_{s-1}$ and $G = \xi_{E/E_{s-1}}/e - A$, then

$$\Theta_{E/E_{s-1}} - f_{E/E_{s-1}}^*(A) = \Theta_{E/E_{s-1}} - f_{E/E_{s-1}}^*(\xi_{E/E_{s-1}}/e) + f_{E/E_{s-1}}^*(G).$$

Since $\Theta_{E/E_{s-1}} - f_{E/E_{s-1}}^*(\xi_{E/E_{s-1}}/e)$ is nef by (F1) and

$$\left(\Theta_{E/E_{s-1}} - f_{E/E_{s-1}}^*(\xi_{E/E_{s-1}}/e)\right)^e = 0,$$

we have

$$0 \leq \left(\Theta_{E/E_{s-1}} - f_{E/E_{s-1}}^*(A)\right)^e = e\deg(G).$$

Therefore, $\deg(G) \geq 0$, and hence $\deg(A) \leq \mu(E/E_{s-1})$.

(d) We can find an irreducible curve C_0 of X such that $(\Theta_E - f_E^*(A) \cdot C_0) < 0$. Clearly C_0 is flat over C. Let C_1 be the normalization of C_0 and $h : C_1 \to C$ the induced morphism. Let us consider the following commutative diagram:

$$\begin{tikzcd} \mathbb{P}(E) \arrow[d, "f_E"'] & \mathbb{P}(h^*(E)) \arrow[l, "\mathbb{P}(h)"'] \arrow[d, "f_{h^*(E)}"] \\ C & C_1 \arrow[l, "h"'] \end{tikzcd}$$

Note that $\mathbb{P}(h)^*(\Theta_E - f_E^*(A)) \sim_{\mathbb{R}} \Theta_{h^*(E)} - f_{h^*(E)}^*(h^*(A))$. Further, there is a section S of $f_{h^*(E)}$ such that $\mathbb{P}(h)_*(S) = C_0$. Let Q be the quotient line bundle of $h^*(E)$ corresponding to the section S. As

$$0 = h^*(E_0) \subsetneq h^*(E_1) \subsetneq h^*(E_2) \subsetneq \cdots \subsetneq h^*(E_{s-1}) \subsetneq h^*(E_s) = h^*(E)$$

is the Harder-Narasimham filtration of $h^*(E)$, we can easily see

$$\deg(Q) \geq \mu(h^*(E/E_{s-1})) = \deg(h)\mu(E/E_{s-1}).$$

On the other hand,

$$\deg(Q) - \deg(h)\deg(A) = (\Theta_{h^*(E)} - f_{h^*(E)}^*(h^*(A)) \cdot S) = (\Theta_E - f_E^*(A) \cdot C_0) < 0,$$

and hence $\mu(E/E_{s-1}) < \deg(A)$. □

Finally let us consider the following three results.

Lemma 1.3. *Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$. Let $\mu : X' \to X$ be a generically finite morphism of normal projective varieties over k. For a $\mathbb{K}$-Cartier divisor D on X, D is $\mathbb{K}$-effective if and only if $\mu^*(D)$ is $\mathbb{K}$-effective.*

Proof. Clearly, if D is $\mathbb{K}$-effective, then $\mu^*(D)$ is $\mathbb{K}$-effective. Let K and K' be the function fields of X and X', respectively. Here we assume that $\mu^*(D)$ is $\mathbb{K}$-effective, that is, there are $\phi'_1, \ldots, \phi'_r \in K'^{\times}$ and $a_1, \ldots, a_r \in \mathbb{K}$ such that $\mu^*(D) + a_1(\phi'_1) + \cdots + a_r(\phi'_r)$ is effective, so that

$$\mu_*\left(\mu^*(D) + a_1(\phi'_1) + \cdots + a_r(\phi'_r)\right) = \deg(\mu)D + a_1\mu_*((\phi'_1)) + \cdots + a_r\mu_*((\phi'_r))$$

is effective. Note that $\mu_*(\phi'_i) = (N_{K'/K}(\phi'_i))$ (cf. [7, Proposition 1.4]), where $N_{K'/K}$ is the norm map of K' over K, and hence

$$D + (a_1/\deg(\mu))(N_{K'/K}(\phi'_1)) + \cdots + (a_r/\deg(\mu))(N_{K'/K}(\phi'_r))$$

is effective. Therefore, D is $\mathbb{K}$-effective. □

Lemma 1.4. *Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$. We assume that k is an algebraic closure of a finite field. Let X be a normal projective variety over k and D a $\mathbb{K}$-Cartier divisor on X. If D is numerically trivial, then D is $\mathbb{K}$-linearly equivalent to the zero divisor.*

Proof. If $\mathbb{K} = \mathbb{Q}$, then the assertion is well-known, so that we assume that $\mathbb{K} = \mathbb{R}$. We set $D = a_1D_1 + \cdots + a_rD_r$, where $D_1, \ldots, D_r$ are Cartier divisors on X and $a_1, \ldots, a_r \in \mathbb{R}$. Considering a $\mathbb{Q}$-basis of $\mathbb{Q}a_1 + \cdots + \mathbb{Q}a_r$ in $\mathbb{R}$, we may assume that $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. Let C be an irreducible curve on X. Note that

$$0 = (D \cdot C) = a_1(D_1 \cdot C) + \cdots + a_r(D_r \cdot C)$$

and $(D_1 \cdot C), \ldots, (D_r \cdot C) \in \mathbb{Z}$, and hence $(D_1 \cdot C) = \cdots = (D_r \cdot C) = 0$ because $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. Thus $D_1, \ldots, D_r$ are numerically equivalent to zero, so that $D_1, \ldots, D_r$ are $\mathbb{Q}$-linearly equivalent to the zero divisor. Therefore, the assertion follows. □

Proposition 1.5. *Let X be a normal projective variety over k and let D be a $\mathbb{Q}$-Cartier divisor on X. If D is $\mathbb{R}$-effective, then D is $\mathbb{Q}$-effective.*

Proof. As D is $\mathbb{R}$-effective, there are non-zero rational functions $\psi_1, \ldots, \psi_l$ on X and $b_1, \ldots, b_l \in \mathbb{R}$ such that $D + b_1(\psi_1) + \cdots + b_l(\psi_l)$ is effective. We set $V = \mathbb{Q}b_1 + \cdots + \mathbb{Q}b_l \subseteq \mathbb{R}$. If $V \subseteq \mathbb{Q}$, then $b_1, \ldots, b_l \in \mathbb{Q}$, so that we may assume that $V \not\subseteq \mathbb{Q}$.

Claim 1.5.1. *There are non-zero rational functions $\phi_1, \ldots, \phi_r$ on X, $a_1, \ldots, a_r \in \mathbb{R}$ and a $\mathbb{Q}$-Cartier divisor D' on X such that $D \sim_{\mathbb{Q}} D'$, $D' + a_1(\phi_1) + \cdots + a_r(\phi_r)$ is effective and $1, a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$.*

Proof. We can find a basis $a_1, \ldots, a_r$ of V over $\mathbb{Q}$ with the following properties:

(i) If we set $b_i = \sum_{j=1}^{r} c_{ij} a_j$, then $c_{ij} \in \mathbb{Z}$ for all i, j.
(ii) If $V \cap \mathbb{Q} \neq \{0\}$, then $a_1 \in \mathbb{Q}^{\times}$.

We put $\phi_j = \prod_{i=1}^{l} \psi_i^{c_{ij}}$. Note that $\sum_{i=1}^{l} b_i(\psi_i) = \sum_{j=1}^{r} a_j(\phi_j)$. Therefore, in the case where $V \cap \mathbb{Q} = \{0\}$, $1, a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$ and $D + \sum_{j=1}^{r} a_j(\phi_j)$ is effective. Otherwise, $1, a_2, \ldots, a_r$ are linearly independent over $\mathbb{Q}$ and $(D + a_1(\phi_1)) + \sum_{j=2}^{r} a_j(\phi_j)$ is effective. □

We set $L = D' + a_1(\phi_1) + \cdots + a_r(\phi_r)$. Let Γ be a prime divisor with $\Gamma \not\subseteq \operatorname{Supp}(L)$. Then

$$0 = \operatorname{mult}_\Gamma(L) = \operatorname{mult}_\Gamma(D') + a_1 \operatorname{ord}_\Gamma(\phi_1) + \cdots + a_r \operatorname{ord}_\Gamma(\phi_r),$$

so that $\operatorname{mult}_\Gamma(D') = \operatorname{ord}_\Gamma(\phi_1) = \cdots = \operatorname{ord}_\Gamma(\phi_r) = 0$ because $1, a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. Thus,

$$\operatorname{Supp}(D'), \operatorname{Supp}((\phi_1)), \ldots, \operatorname{Supp}((\phi_r)) \subseteq \operatorname{Supp}(L).$$

Therefore, we can find $a'_1, \ldots, a'_r \in \mathbb{Q}$ such that $D' + a'_1(\phi_1) + \cdots + a'_r(\phi_r)$ is effective, and hence D is $\mathbb{Q}$-effective. □

2. Proof of Theorem 0.4

Let k be an algebraic closure of a finite field. Let C be a smooth projective curve over k. Let us begin with the following lemma.

Lemma 2.1. *Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$. Let A be a $\mathbb{K}$-Cartier divisor on C. If $\deg(A) \geq 0$, then A is $\mathbb{K}$-effective.*

Proof. If $\mathbb{K} = \mathbb{Q}$, then the assertion is obvious. We assume that $\mathbb{K} = \mathbb{R}$. If $\deg(A) = 0$, the assertion follows from Lemma 1.4. Next we consider the case where $\deg(A) > 0$. We can find a $\mathbb{Q}$-Cartier divisor A' such that $A' \leq A$ and $\deg(A') > 0$. Thus the previous observation implies the assertion. □

As a consequence of (F3), (F4) and (F5), we have the following splitting theorem, which was obtained by Biswas and Parameswaran [2, Proposition 2.1].

Theorem 2.2. *For a locally free sheaf E on C, there are a surjective morphism $\pi : C' \to C$ of smooth projective curves over k and invertible sheaves $L_1, \ldots, L_r$ on C' such that $\pi^*(E) \simeq L_1 \oplus \cdots \oplus L_r$.*

Proof. For reader's convenience, we give a sketch of the proof. First we assume that E is strongly semistable. Let ξ_E be a Cartier divisor on C with $\mathcal{O}_C(\xi_E) \simeq \det(E)$. Let $h : B \to C$ be a surjective morphism of smooth projective curves over k such that $h^*(\xi_E)$ is divisible by $\operatorname{rk}(E)$. We set $E' = h^*(E) \otimes \mathcal{O}_B(-h^*(\xi_E)/\operatorname{rk}(E))$. As $\det(E') \simeq \mathcal{O}_B$, the assertion follows from (F5).

By the above observation, it is sufficient to find a surjective morphism $\pi : C' \to C$ of smooth projective curves over k and strongly semistable locally free sheaves $Q_1, \ldots, Q_n$ on C' such that

$$\pi^*(E) = Q_1 \oplus \cdots \oplus Q_n.$$

Moreover, by (F4), we may assume that E has the strong Harder-Narasimham filtration

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{n-1} \subsetneq E_n = E.$$

Clearly we may further assume that $n \geq 2$. For a non-negative integer m, we set

$$C_m := X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(k),$$

where the morphism $\operatorname{Spec}(k) \to \operatorname{Spec}(k)$ is given by $x \mapsto x^{1/p^m}$. Let $F_k^m : C_m \to C$ be the relative m-th Frobenius morphism over k. Put

$$G_{i,j}^m := (F_k^m)^* \left((E_j/E_i) \otimes (E_i/E_{i-1})^\vee \right) \otimes \omega_{C_m}$$

for $i = 1, \ldots, n-1$ and $j = i, \ldots, n$. We can find a positive integer m such that

$$\mu\left(G_{i,i+1}^m\right) = p^m \left(\mu(E_{i+1}/E_i) - \mu(E_i/E_{i-1})\right) + \deg(\omega_C) < 0$$

for all $i = 1, \ldots, n-1$. By using (F3), we can see that

$$0 = G_{i,i}^m \subsetneq G_{i,i+1}^m \subsetneq G_{i,i+2}^m \subsetneq \cdots \subsetneq G_{i,n-1}^m \subsetneq G_{i,n}^m$$

is the strong Harder-Narasimham filtration of $G_{i,n}^m$, so that $H^0\left(C_m, G_{i,n}^m\right) = \{0\}$, which yields

$$\operatorname{Ext}^1\left((F_k^m)^*(E/E_i), (F_k^m)^*(E_i/E_{i-1})\right) = 0$$

because of Serre's duality theorem. Therefore, an exact sequence

$$0 \to (F_k^m)^*(E_i/E_{i-1}) \to (F_k^m)^*(E/E_{i-1}) \to (F_k^m)^*(E/E_i) \to 0$$

splits, that is, $(F_k^m)^*(E/E_{i-1}) \simeq (F_k^m)^*(E_i/E_{i-1}) \oplus (F_k^m)^*(E/E_i)$ for $i = 1, \ldots, n-1$, and hence

$$(F_k^m)^*(E) \simeq \bigoplus_{i=1}^n (F_k^m)^*(E_i/E_{i-1}),$$

as required. □

Proof of Theorem 0.4. By virtue of Theorem 2.2 and Lemma 1.3, we may assume that

$$E \simeq L_1 \oplus \cdots \oplus L_r$$

for some invertible sheaves $L_1, \ldots, L_r$ on C. We set

$$d = \max\{\deg(L_1), \ldots, \deg(L_r)\} \quad \text{and} \quad I = \{i \mid \deg(L_i) = d\}.$$

There is a $\mathbb{K}$-Cartier divisor A on C such that $D \sim_{\mathbb{K}} \lambda\Theta_E - f_E^*(A)$ for some $\lambda \in \mathbb{K}$. Let M be an ample divisor on C such that $T := \Theta_E + f_E^*(M)$ is ample. As D is pseudo-effective, we have

$$0 \leq (D \cdot T^{r-2} \cdot f_E^*(M)) = ((\lambda T - f_E^*(A + \lambda M)) \cdot T^{r-2} \cdot f_E^*(M)) = \lambda \deg(M),$$

and hence $\lambda \geq 0$. If $\lambda = 0$, then $0 \leq (D \cdot T^{r-1}) = \deg(-A)$. Thus, by Lemma 2.1, $-A$ is $\mathbb{K}$-effective, so that the assertion follows.

We assume that $\lambda > 0$. Replacing D by D/λ, we may assume that $\lambda = 1$. Let ξ be a Cartier divisor on C such that $\mathcal{O}_C(\xi) \simeq L_{i_0}$ for some $i_0 \in I$. Note that the first part E_1 of the strong Harder-Narasimham filtration of E is $\bigoplus_{i \in I} L_i$, so that, by Proposition 1.1, $\deg(A) \leq \deg(\xi)$. If we set $B = \xi - A$, then, by Lemma 2.1, B is $\mathbb{K}$-effective because $\deg(B) \geq 0$. Moreover, as

$$\Theta_E - f_E^*(A) = \Theta_E - f_E^*(\xi) + f_E^*(B),$$

it is sufficient to consider the case where $D = \Theta_E - f_E^*(\xi)$. In this case, the assertion is obvious because

$$H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(D)) = H^0(C, E \otimes \mathcal{O}_C(-\xi)) = H^0\left(C, \bigoplus_{i=1}^{r} L_i \otimes \mathcal{O}_C(-\xi)\right) \neq \{0\}.$$

□

As a consequence of Theorem 0.4, we can recover a result due to [3].

Corollary 2.3. *Let k, C and E be same as in Theorem 0.4. We assume that $r = 2$. Let D be a Cartier divisor on $\mathbb{P}(E)$ such that $(D \cdot Y) > 0$ for all irreducible curves Y on $\mathbb{P}(E)$. Then D is ample.*

Proof. As D is nef, D is pseudo-effective, so that, by Theorem 0.4, there is an effective $\mathbb{Q}$-Cartier divisor E on X such that $D \sim_{\mathbb{Q}} E$. As $E \neq 0$, we have $(D \cdot D) = (D \cdot E) > 0$. Therefore, D is ample by Nakai-Moishezon criterion. □

Remark 2.4. The argument in the proof of Corollary 2.3 actually shows that the $\mathbb{Q}$-version of Question 0.2 on algebraic surfaces implies Question 0.3.

3. Numerically effectivity on abelian varieties

The purpose of this section is to give an affirmative answer for the $\mathbb{Q}$-version of Question 0.2 on abelian varieties. Let A be an abelian variety over an algebraically closed field k. A key observation is the following proposition.

Proposition 3.1. *If a $\mathbb{Q}$-Cartier divisor D on A is nef, then D is numerically equivalent to a $\mathbb{Q}$-effective $\mathbb{Q}$-Cartier divisor.*

Proof. We prove it by induction on $\dim A$. If $\dim A \leq 1$, then the assertion is obvious. Clearly we may assume that D is a Cartier divisor, so that we set $L = \mathcal{O}_A(D)$. As $L \otimes [-1]^*(L)$ is numerically equivalent to $L^{\otimes 2}$ (cf. [21, p.75, (iv)]), we may assume that L is symmetric, that is, $L \simeq [-1]^*(L)$. Let $K(L)$ be the closed subgroup of A given by $K(L) = \{x \in A \mid T_x^*(L) \simeq L\}$ (cf. [21, p.60, Definition]). If $K(L)$ is finite, then L is nef and big by virtue of [21, p.150, The Riemann-Roch theorem], so that D is $\mathbb{Q}$-effective. Otherwise, let B be the connected component of $K(L)$ containing 0.

Claim 3.1.1. (1) *$T_x^*(L)|_B \simeq L|_B$ for all $x \in A$.*
(2) *$L^{\otimes 2}\big|_{B+x} \simeq \mathcal{O}_{B+x}$ for $x \in A$.*

Proof. (1) Let N be an invertible sheaf on $A \times A$ given by

$$N = m^*(L) \otimes p_1^*(L^{-1}) \otimes p_2^*(L^{-1}),$$

where $p_i : A \times A \to A$ is the projection to the i-th factor ($i = 1, 2$) and m is the addition morphism. Note that $N|_{B\times A} \simeq \mathcal{O}_{B\times A}$ (cf. [21, p.123, §13]). Fixing $x \in A$, let us consider a morphism $\alpha : B \to B \times A$ given by $\alpha(y) = (y, x)$. Then

$$\mathcal{O}_B \simeq \alpha^* \left(m^*(L) \otimes p_1^*(L^{-1}) \otimes p_2^*(L^{-1})\Big|_{B\times A} \right) \simeq T_x^*(L)|_B \otimes L^{-1}\Big|_B,$$

as required.

(2) First we consider the case where $x = 0$. As $N|_{B\times A} \simeq \mathcal{O}_{B\times A}$, we have $N|_{B\times B} \simeq \mathcal{O}_{B\times B}$. Using a morphism $\beta : B \to B \times B$ given by $\beta(y) = (y, -y)$, we have

$$\mathcal{O}_B \simeq \beta^*(N|_{B\times B}) = L^{-1}\Big|_B \otimes [-1]^*(L^{-1})\Big|_B \simeq L^{\otimes -2}\Big|_B,$$

as required.

In general, for $x \in A$, by (1) and the previous observation together with the following commutative diagram

$$\begin{array}{ccc} B + x & \longrightarrow & A \\ {\scriptstyle T_{-x}}\big\downarrow & & \big\downarrow{\scriptstyle T_{-x}} \\ B & \longrightarrow & A, \end{array}$$

we can see

$$\begin{aligned} \mathcal{O}_{B+x} = T_{-x}^*(\mathcal{O}_B) &\simeq T_{-x}^*\left(L^{\otimes 2}\Big|_B\right) \simeq T_{-x}^*\left(T_x^*(L)^{\otimes 2}\Big|_B\right) \\ &= T_{-x}^*\left(T_x^*(L^{\otimes 2})\Big|_B\right) = T_{-x}^*(T_x^*(L^{\otimes 2}))\Big|_{B+x} = L^{\otimes 2}\Big|_{B+x}. \end{aligned}$$

□

Let $\pi : A \to A/B$ be the canonical homomorphism. By (2) in the above claim,

$$\dim_{k(y)} H^0\left(\pi^{-1}(y), L^{\otimes 2}\right) = 1$$

for all $y \in A/B$, so that, by [21, p.51, Corollary 2], $\pi_*(L^{\otimes 2})$ is an invertible sheaf on A/B and $\pi_*(L^{\otimes 2}) \otimes k(y) \xrightarrow{\sim} H^0(\pi^{-1}(y), L^{\otimes 2})$. Therefore, the natural homomorphism $\pi^*(\pi_*(L^{\otimes 2})) \to L^{\otimes 2}$ is an isomorphism, that is, there is a $\mathbb{Q}$-Cartier divisor D' on A/B such that $\pi^*(D') \sim_{\mathbb{Q}} D$. Note that D' is also nef, so that, by the hypothesis of induction, D' is numerically equivalent to a $\mathbb{Q}$-effective $\mathbb{Q}$-Cartier divisor, and hence the assertion follows. □

Proof of Proposition 0.5. Proposition 0.5 is a consequence of Lemma 1.4 and Proposition 3.1 because a pseudo-effective $\mathbb{Q}$-Cartier divisor on an abelian variety is nef. □

Example 3.2. Here we show that the $\mathbb{R}$-version of Question 0.2 does not hold in general. Let k be an algebraically closed field (k is not necessarily an algebraic closure of a finite field). Let C be an elliptic curve over k and $A := C \times C$. Let $\mathrm{NS}(A)$ be the Néron-Severi group of A. Note that $\rho := \operatorname{rk} \mathrm{NS}(A) \geq 3$. By using the Hodge index theorem, we can find a basis $e_1, \ldots, e_\rho$ of $\mathrm{NS}(A)_{\mathbb{Q}} := \mathrm{NS}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ with the following properties:

(1) e_1 is the class of the divisor $\{0\} \times C + C \times \{0\}$. In particular, $(e_1 \cdot e_1) = 2$.
(2) $(e_i \cdot e_i) < 0$ for all $i = 2, \ldots, \rho$.
(3) $(e_i \cdot e_j) = 0$ for all $1 \leq i \neq j \leq \rho$.

We set $\lambda_i := -(e_i \cdot e_i)$ for $i = 2, \ldots, \rho$. Let $\overline{\mathrm{Amp}}(A)$ be the closed cone in $\mathrm{NS}(A)_{\mathbb{R}} := \mathrm{NS}(A) \otimes_{\mathbb{Z}} \mathbb{R}$ generated by ample $\mathbb{Q}$-Cartier divisors on A. It is well known that

$$\begin{aligned}\overline{\mathrm{Amp}}(A) &= \left\{\xi \in \mathrm{NS}(A)_{\mathbb{R}} \mid (\xi^2) \geq 0,\ (\xi \cdot e_1) \geq 0\right\} \\ &= \left\{x_1 e_1 + x_2 e_2 + \cdots + x_\rho e_\rho \mid \lambda_2 x_2^2 + \cdots + \lambda_\rho x_\rho^2 \leq 2x_1^2,\ x_1 \geq 0\right\}.\end{aligned}$$

We choose $(a_2, \ldots, a_\rho) \in \mathbb{R}^{\rho-1}$ such that

$$(a_2, \ldots, a_\rho) \notin \mathbb{Q}^{\rho-1} \quad \text{and} \quad \lambda_2 a_2^2 + \cdots + \lambda_\rho a_\rho^2 = 2.$$

Let E_i be a $\mathbb{Q}$-Cartier divisor on A such that the class of E_i in $\mathrm{NS}(A)_{\mathbb{Q}}$ is equal to e_i for $i = 1, \ldots, \rho$. If we set $D := E_1 + a_2 E_2 + \cdots + a_\rho E_\rho$, then we have the following claim, which is sufficient for our purpose.

Claim 3.2.1. *D is nef and D is not numerically equivalent to an effective $\mathbb{R}$-Cartier divisor.*

Proof. Clearly D is nef. If we set $e_1' = e_1/\sqrt{2}$ and $e_i' = e_i/\sqrt{\lambda_i}$ for $i = 2, \ldots, \rho$, then

$$\overline{\mathrm{Amp}}(A) = \left\{y_1 e_1' + y_2 e_2' + \cdots + y_\rho e_\rho' \mid {y_2}^2 + \cdots + {y_\rho}^2 \leq {y_1}^2,\ y_1 \geq 0\right\}.$$

Therefore, as $[D] \in \partial(\overline{\mathrm{Amp}}(A)_{\mathbb{R}})$, we can choose

$$H \in \mathrm{Hom}_{\mathbb{R}}(\mathrm{NS}(A)_{\mathbb{R}}, \mathbb{R})$$

such that

$$H \geq 0 \text{ on } \overline{\mathrm{Amp}}(A) \quad \text{and} \quad \{H = 0\} \cap \overline{\mathrm{Amp}}(A) = \mathbb{R}_{\geq 0}[D],$$

where $[D]$ is the class of D in $\mathrm{NS}(A)_{\mathbb{R}}$. We assume that D is numerically equivalent to an effective $\mathbb{R}$-Cartier divisor $c_1\Gamma_1 + \cdots + c_r\Gamma_r$, where $c_1, \ldots, c_r \in \mathbb{R}_{\geq 0}$ and $\Gamma_1, \ldots, \Gamma_r$ are prime divisors on A. As $[D] \neq 0$, we may assume that $c_1, \ldots, c_r \in \mathbb{R}_{>0}$. Note that $[\Gamma_1], \ldots, [\Gamma_r] \in \overline{\mathrm{Amp}}(A)$ and

$$0 = H([D]) = c_1 H([\Gamma_1]) + \cdots + c_r H([\Gamma_r]),$$

so that $H([\Gamma_1]) = \cdots = H([\Gamma_r]) = 0$, and hence $[\Gamma_1], \ldots, [\Gamma_r] \in \mathbb{R}_{\geq 0}[D]$. In particular, there is $t \in \mathbb{R}_{\geq 0}$ with $[\Gamma_1] = t[D]$. Here we can set

$$[\Gamma_1] = b_1 e_1 + \cdots + b_\rho e_\rho \quad (b_1, \ldots, b_\rho \in \mathbb{Q}).$$

Thus $b_1 = t, b_2 = ta_2, \ldots, b_\rho = ta_\rho$. As $[\Gamma_1] \neq 0$, $t \in \mathbb{Q}^\times$, and hence $(a_2, \ldots, a_\rho) = t^{-1}(b_2, \ldots, b_\rho) \in \mathbb{Q}^{\rho-1}$. This is a contradiction. $\square$

Remark 3.3. Let k be an algebraic closure of a finite field and let X be a normal projective variety over k. Let $\mathrm{NS}(X)$ be the Néron-Severi group of X and $\mathrm{NS}(X)_{\mathbb{R}} := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\overline{\mathrm{Eff}}(X)$ be the closed cone in $\mathrm{NS}(X)_{\mathbb{R}}$ generated by pseudo-effective $\mathbb{R}$-Cartier divisors on X. We assume that $\overline{\mathrm{Eff}}(X)$ is a rational polyhedral cone, that is, there are pseudo-effective $\mathbb{Q}$-Cartier divisors $D_1, \ldots, D_n$ on X such that $\overline{\mathrm{Eff}}(X)$ is generated by the classes of $D_1, \ldots, D_n$. Then the $\mathbb{Q}$-version of Question 0.2 implies the $\mathbb{R}$-version of Question 0.2.

Example 3.4. This is an example due to Yuan [25]. Let us fix an algebraically closed field k and an integer $g \geq 2$. Let C be a smooth projective curve over k and $f : X \to C$ an abelian scheme over C of relative dimension g. Let L be an f-ample invertible sheaf on X such that $[-1]^*(L) \simeq L$ and L is trivial along the zero section of $f : X \to C$.

Claim 3.4.1. (1) $[2]^*(L) \simeq L^{\otimes 4}$.
(2) *L is nef.*

Proof. (1) As $[2]^*(L)|_{f^{-1}(x)} \simeq L^{\otimes 4}|_{f^{-1}(x)}$ for all $x \in C$, there is an invertible sheaf M on C such that $[2]^*(L) \simeq L^{\otimes 4} \otimes f^*(M)$. Let Z_0 be the zero section of $f : X \to C$. Then

$$\mathcal{O}_{Z_0} \simeq [2]^*(L|_{Z_0}) = [2]^*(L)|_{Z_0} \simeq L^{\otimes 4} \otimes f^*(M)\Big|_{Z_0} \simeq M,$$

so that we have the assertion.

(2) Let A be an ample invertible sheaf on C such that $L \otimes f^*(A)$ is ample. Let Δ be a horizontal curve on X. As $f \circ [2^n] = f$ and $[2^n]^*(L) \simeq L^{\otimes 4^n}$ by using (1),

$$0 \leq (L \otimes f^*(A) \cdot [2^n]_*(\Delta)) = ([2^n]^*(L \otimes f^*(A)) \cdot \Delta) = (L^{\otimes 4^n} \otimes f^*(A) \cdot \Delta),$$

so that $(L \cdot \Delta) \geq -4^{-n}(f^*(A) \cdot \Delta)$ for all $n > 0$. Thus $(L \cdot \Delta) \geq 0$. $\square$

Claim 3.4.2. *If the characteristic of k is zero and f is non-isotrivial, then L does not have the Dirichlet property* (*i.e. L is not $\mathbb{Q}$-effective*).

Proof. The following proof is due to Yuan [25]. An alternative proof can be found in [6, Theorem 4.3]. We need to see that $H^0(X, L^{\otimes n}) = 0$ for all $n > 0$. We set $d_n = \mathrm{rk}\, f_*(L^{\otimes n})$. By changing the base C if necessarily, we may assume that all $(d_n)^2$-torsion points on the generic fiber X_η of $f : X \to C$ are defined over the function field of C. By using the algebraic theta theory due to Mumford (especially [20, the last line in page 81]), there is an invertible sheaf M on C such that $f_*(L^{\otimes n}) = M^{\oplus d_n}$. On the other hand, by [13],

$$\deg(\det(f_*(L^{\otimes n}))^{\otimes 2} \otimes f_*(\omega_{X/C})^{\otimes d_n}) = 0,$$

that is, $2\deg(M) + \deg(f_*(\omega_{X/C})) = 0$. As f is non-isotrivial, we can see that $\deg(f_*(\omega_{X/C})) > 0$, so that $\deg(M) < 0$, and hence the assertion follows. $\square$

If the characteristic of k is positive, we do not know the $\mathbb{Q}$-effectivity of L in general. In [15], there is an example with the following properties:

(1) $g = 2$ and $C = \mathbb{P}^1_k$.
(2) There are an abelian surface A over k and an isogeny $h : A \times \mathbb{P}^1_k \to X$ over $\mathbb{P}^1_k$.

Claim 3.4.3. *In the above example, L has the Dirichlet property.*

Proof. Replacing L by $L^{\otimes n}$, we may assume that $d := \operatorname{rk} f_*(L) > 0$. Let

$$p_1 : A \times \mathbb{P}^1_k \to A \quad \text{and} \quad p_2 : A \times \mathbb{P}^1_k \to \mathbb{P}^1_k$$

be the projections to A and $\mathbb{P}^1_k$, respectively. Note that $h^*(L)$ is symmetric and $h^*(L)$ is trivial along the zero section of p_2. Since $\omega_{A\times\mathbb{P}^1_k/P^1_k} \simeq p_1^*(\omega_A)$, we have $(p_2)_*(\omega_{A\times\mathbb{P}^1_k/P^1_k}) \simeq \mathcal{O}_{\mathbb{P}^1_k}$, so that, by [13], $\deg(\det((p_2)_*(h^*(L)))) = 0$, that is, if we set

$$(p_2)_*(h^*(L)) = \mathcal{O}_{\mathbb{P}^1_k}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_d),$$

then $a_1 + \cdots + a_d = 0$. Thus $a_i \geq 0$ for some i, and hence

$$H^0(A \times \mathbb{P}^1_k, h^*(L)) \neq 0.$$

Therefore, L is $\mathbb{Q}$-effective by Lemma 1.3. □

The above claim suggests that the set of preperiodic points of the map $[2] : X \to X$ is not dense in the analytification X_v^{an} at any place v of $\mathbb{P}^1_k$ with respect to the analytic topology (cf. [5]).

References

[1] I. Biswas and Y. I. Holla, Comparison of fundamental group schemes of a projective variety and an ample hypersurface, J. Alg. Geom. **16** (2007), 547–597.

[2] I. Biswas and A.J. Parameswaran, On vector bundles on curves over $\overline{\mathbb{F}}_p$, C. R. Acad. Sci. Paris, Ser. I, **350** (2012), 213–216.

[3] I. Biswas and S. Subramanian, On a question of Sean Keel, J. of Pure and Applied Algebra, **215** (2011), 2600–2602.

[4] J. I. Burgos Gil, A. Moriwaki, P. Philippon and M. Sombra, Arithmetic positivity on toric varieties, to appear in J. of Alg. Geom., (see also arXiv:1210.7692v1 [math.AG]).

[5] H. Chen and A. Moriwaki, Algebraic dynamical systems and Dirichlet's unit theorem on arithmetic varieties, preprint, arXiv:1312.3528v4 [math.AG].

[6] E. Colombo and G. P. Pirola, An intersection result for families of abelian varieties, J. of Pure and Applied Algebra, **129** (1998), 111-122.

[7] W. Fulton, *Intersection Theory*, Second Edition, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag (1998).

[8] R. Hartshorne, *Ample subvarieties of algebraic varieties*, Lecture Notes in Mathematics, No. **145**. Springer, New York, (1970).

[9] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. **52**. Springer, New York, (1977).

[10] S. Keel, Polarized pushouts over finite fields, Comm. Alg., **31** (2003), 3955–3982.

[11] A. Langer, Semistable sheaves in positive characteristic, Ann. of Math., **159** (2004), 252–276.

[12] A. Langer, On positivity and semistability of vector bundles in finite and mixed characteristics, J. Ramanujan Math. Soc., **28A** (2013), 287–309.

[13] V. Maillot, D. Rössler, On the determinant bundles of abelian schemes. Compos. Math., **144** (2008), no. 2, 495–502.

[14] V. Mehta and S. Subramanian, Nef line bundles which are not ample, Math. Zeit., **219** (1995), 235-244.

[15] L. Moret-Bailly, Familles de courbes et de variétés abéliennes sur $\mathbb{P}^1$, In: L. Szpiro (ed.), Séminaire sur les pinceaux de courbes de genre au moins deux, Astérisque **86** (1981).
[16] A. Moriwaki, Relative Bogomolov's inequality and the cone of positive divisors on the moduli space of stable curves, J. Amer. Math. Soc. **11** (1998), no. 3, 569–600.
[17] A. Moriwaki, Zariski decompositions on arithmetic surfaces, Publ. Res. Inst. Math. Sci. **48** (2012), 799-898.
[18] A. Moriwaki, Toward Dirichlet's unit theorem on arithmetic varieties, Kyoto J. of Math., **53** (2013), 197–259.
[19] A. Moriwaki, Adelic divisors on arithmetic varieties, preprint (arXiv:1302.1922 [math.AG]).
[20] D. Mumford, On the equations defining abelian varieties. II. Invent. Math. 3 (1967) 75–135.
[21] D. Mumford, *Abelian Varieties*, Oxford University Press (1974).
[22] N. Nakayama, Zariski-decomposition and Abundance, MSJ Memoirs, **14** (2004), Mathematical Society of Japan.
[23] S. Subramanian, Strongly semistable bundles on a curve over a finite field, Arch. Math. **89** (2007), 68–72.
[24] B. Totaro, Moving codimension-one subvarieties over finite fields, Amer. J. Math., **131** (2009), 1815-1833.
[25] X. Yuan, A nef but non-effective line bundle, private note.

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